

Willmore-like functionals for surfaces in 3-dimensional Thurston geometries

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ABSTRACT. We find analogues of the Willmore functional for each of the Thurston geometries with 4-dimensional isometry group such that the CMC-spheres in these geometries are critical points of these functionals.

1 Introduction

Let M be a closed orientable surface and $f: M \rightarrow N$ be an immersion of M into a 3-dimensional Riemannian manifold N . Set:

$$\mathcal{W}(f) = \int_M (H^2 + \overline{K}) d\mu,$$

where H is the mean curvature of the immersed surface, the value of \overline{K} at a point $p \in M$ is defined as the sectional curvature of the 2-plane $f_*(T_p M)$ in N , $d\mu$ is the area element of the induced metric on M . We will refer to the functional \mathcal{W} as the Willmore functional. It is known that $\mathcal{W}(f)$ is a conformal invariant [1].

In the 3-dimensional space forms $\mathbb{R}^3, \mathbb{H}^3$ and \mathbb{S}^3 the functional \mathcal{W} enjoys the property that the CMC spheres are critical points of \mathcal{W} ; recall that in the 3-dimensional space forms the CMC spheres are exactly the round spheres by the Hopf theorem. However, this property for the Willmore functional \mathcal{W} fails to hold in the other 3-dimensional Thurston geometries.

In this paper we will introduce the Willmore-like functionals for the certain family of Riemannian manifolds $E(k, \tau)$ that include the model spaces for all Thurston geometries with 4-dimensional group of isometries, i.e., the products $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$, the Heisenberg group Nil and the Lie group $\widetilde{\text{PSL}_2(\mathbb{R})}$. The functionals to be introduced in these geometries have the form:

$$\int_M (H^2 + \alpha \overline{K} + \beta) d\mu, \tag{1}$$

where α and β are some constants that depend on k and τ . In the case of the Heisenberg group Nil (for $k = 0$ and $\tau = \frac{1}{2}$) the functional:

$$\int_M \left(H^2 + \frac{1}{4} \overline{K} - \frac{1}{16} \right) d\mu$$

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was obtained in [2] based on the Weierstrass representation for surfaces in Nil. Then it was shown [3] that the CMC spheres in Nil are critical points of this functional. In the case of the Lie group $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$ (for $k = -1$ and $\tau = -\frac{1}{2}$) it was shown that for the functional:

$$\int_M \left(H^2 + \frac{1}{4}\overline{K} - \frac{5}{16} \right) d\mu$$

the minimum among the rotationally invariant spheres is attained exactly at the CMC spheres [4].

The main result of the paper is the following theorem.

Theorem 1. *The CMC spheres in $E(k, \tau)$ are critical points of the following Willmore-like functional:*

$$E(f) = \int_M \left(H^2 + \frac{1}{4}\overline{K} + \frac{k}{4} - \frac{\tau^2}{4} \right) d\mu. \quad (2)$$

In addition to Theorem 1 we will prove the following theorem.

Theorem 2. *For rotationally invariant spheres in $E(k, \tau)$ the functional $E(f)$ attains its minimum exactly at the CMC spheres.*

REMARK 1. We note that:

$$E(f) = \mathcal{W}(f) + \int_M \left(-\frac{3}{4}\overline{K} + \frac{k}{4} - \frac{\tau^2}{4} \right) d\mu.$$

REMARK 2. It can be seen that Theorem 1 agrees with the results obtained earlier for Nil [3] and the Lie group $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$ [4]. In addition, the functional E for the case $\mathbb{S}^2 \times \mathbb{R}$ ($k = 1$ and $\tau = 0$) coincides up to a constant factor with the functional $^1 \int_M (4H^2 + \overline{K} + 1)$ mentioned in § 6.2, [3].

The structure of the remaining part of this paper is as follows. In § 2 we give the description of the Riemannian manifolds $E(k, \tau)$. In § 3 we review the characterizations of the CMC spheres in these manifolds. In § 4 we give the details of the proof of Theorem 1. In § 5 we give the details of the proof of Theorem 2.

2 The Riemannian manifolds $E(k, \tau)$

The model spaces for the four Thurston geometries: Nil, $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$, $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ belong to the family of Riemannian 3-manifolds $E(k, \tau)$, $k \in \mathbb{R}$, $\tau \in \mathbb{R}$ that are as follows. If $k \geq 0$ then $E(k, \tau)$ is \mathbb{R}^3 with the metric:

$$ds^2 = \frac{dx^2 + dy^2}{\left(1 + \frac{k}{4}(x^2 + y^2)\right)^2} + \left(dz + \frac{\tau(ydx - xdy)}{1 + \frac{k}{4}(x^2 + y^2)} \right)^2. \quad (3)$$

¹In § 6.2, [3] the term H should be considered as $2H$.

If $k < 0$ then $E(k, \tau)$ is the product $D^2(\frac{2}{\sqrt{-k}}) \times \mathbb{R}$ with the metric (3), where $D^2(\frac{2}{\sqrt{-k}}) = \{(x, y) \mid x^2 + y^2 < \frac{4}{-k}\}$. The family $E(k, \tau)$ is also referred to as Bianchi–Cartan–Vranceanu family [5, 6]. The projection of $E(k, \tau)$ onto the 2-dimensional domain of constant curvature k given by the map $(x, y, z) \mapsto (x, y)$ is a Riemannian fibration. The fibres of such a fiber bundle are geodesics and its unitary tangent vectors $\frac{\partial}{\partial z}$ form a Killing vector field; this field is also referred to as the vertical vector field. The parameter k is called the base curvature and τ the bundle curvature.

If $k = -1, \tau = 0$ then $E(k, \tau)$ is the product $\mathbb{H}^2 \times \mathbb{R}$. If $k = 1, \tau = 0$ then $E(k, \tau)$ is obtained from the product $\mathbb{S}^2 \times \mathbb{R}$ by removing one fibre. If $k = 0, \tau \neq 0$ then $E(k, \tau)$ is the Heisenberg group Nil with the left-invariant metric determined by the parameter τ . If $k < 0, \tau \neq 0$ then $E(k, \tau)$ is the Lie group $\widetilde{\text{PSL}_2(\mathbb{R})}$ with the left-invariant metric determined by the parameters k and τ . For the case $k > 0, \tau \neq 0$, the manifolds $E(k, \tau)$ are obtained from the covering of the Berger spheres by removing one fibre.

For more details we refer the reader to [7, 8]. We will need the following proposition.

Proposition 1. *The sectional curvature of a 2-plane in $E(k, \tau)$ equals:*

$$\overline{K} = \tau^2 + (k - 4\tau^2)\nu^2, \quad (4)$$

where ν is the scalar product of a unit normal vector to the plane and the vertical vector $\xi = \frac{\partial}{\partial z}$ with respect to the metric (3).

Proof. The identity (4) can be obtained directly from the general formula for the Riemann curvature tensor of $E(k, \tau)$ shown in Proposition 2.1, [8]. \square

3 The CMC spheres in $E(k, \tau)$

The rotationally invariant CMC surfaces in the products $\mathbb{H}^2 \times \mathbb{R}, \mathbb{S}^2 \times \mathbb{R}$ and the Heisenberg group Nil were described in [9, 10] and [11, 12, 13] respectively. The case of the Lie group $\widetilde{\text{PSL}_2(\mathbb{R})}$ and the Berger spheres were studied in [14] and [15]. In order to describe rotationally invariant CMC surfaces in $E(k, \tau)$ we will follow the approach used in [11, 12] for $E(0, \frac{1}{2})$.

For the cylindrical coordinates ρ, θ, z in $E(k, \tau)$ such that $x = \rho \cos \theta, y = \rho \sin \theta, z = z$ the metric (3) has the form:

$$ds^2 = \frac{1}{(1 + \frac{k}{4}\rho^2)^2} d\rho^2 + \frac{\rho^2 + \tau^2\rho^4}{(1 + \frac{k}{4}\rho^2)^2} d\theta^2 - \frac{2\tau\rho^2}{1 + \frac{k}{4}\rho^2} dz d\theta + dz^2. \quad (5)$$

We note that $\rho \in [0, R)$, where $R = \frac{2}{\sqrt{-k}}$ if $k < 0$ and $R = \infty$ if $k \geq 0$.

The group $\text{SO}(2)$ acts on $E(k, \tau)$ by rotations $\theta \mapsto \theta + \text{const}$ around z -axis. The rotations are isometries and the factor-space $E(k, \tau)/\text{SO}(2)$ is the 2-dimensional domain $B(k, \tau) = \{(u, v) \mid u \in [0, R), v \in \mathbb{R}\}$ with the metric:

$$d\tilde{s}^2 = \frac{1}{(1 + \frac{k}{4}u^2)^2} du^2 + \frac{1}{1 + \tau^2 u^2} dv^2, \quad (6)$$

so the projection $E(k, \tau) \rightarrow B(k, \tau)$ is a Riemannian submersion.

For a given rotationally invariant surface we define by $\gamma(s) = (u(s), v(s))$ its projection onto $B(k, \tau)$, where s is a natural parameter with respect to the metric (6). Let σ be the angle between $\dot{\gamma}$ and $\frac{\partial}{\partial u}$. It can be verified (cf. [12], eq. 2) that for the metric (6) the geodesic curvature of $\gamma(s)$ equals:

$$\tilde{k} = \dot{\sigma} - \frac{\tau^2 u(1 + \frac{k}{4}u^2)}{(1 + \tau^2 u^2)} \sin \sigma, \quad (7)$$

The mean curvature of a rotationally invariant surface is given by the reduction theorem (cf. [12], p. 178) as follows:

$$H = \frac{1}{2} \left(\tilde{k} - \frac{\partial}{\partial n} \ln \mu \right), \quad (8)$$

where $n = (-(1 + \frac{k}{4}u^2) \sin \sigma, \sqrt{1 + \tau^2 u^2} \cos \sigma)$ is a normal vector in $B(k, \tau)$ to $\gamma(s)$ and $\mu = \frac{u\sqrt{1 + \tau^2 u^2}}{1 + \frac{k}{4}u^2}$ is the factor of the volume form for an $SO(2)$ orbit with respect to the metric (5). From (7) and (8) we obtain:

$$H = \frac{1}{2} \left(\dot{\sigma} + \left(\frac{1}{u} - k \frac{u}{4} \right) \sin \sigma \right). \quad (9)$$

Thus we obtain that for a profile $\gamma(s) = (u(s), v(s))$ of a rotationally invariant CMC surface the following system of ODE is satisfied:

$$\begin{cases} \dot{u} = (1 + \frac{k}{4}u^2) \cos \sigma, \\ \dot{v} = \sqrt{1 + \tau^2 u^2} \sin \sigma, \\ \dot{\sigma} = 2H - \left(\frac{1}{u} - k \frac{u}{4} \right) \sin \sigma. \end{cases} \quad (10)$$

It can be straightforwardly verified that the system (10) has the following first integral:

$$J = \frac{u}{1 + \frac{k}{4}u^2} (\sin \sigma - Hu). \quad (11)$$

Then we have the following proposition.

Proposition 2. *If $k \leq 0$ then for any H such that $H^2 > \frac{-k}{4}$ there exists a rotationally invariant CMC sphere of constant mean curvature H in $E(k, \tau)$; moreover, if $H^2 \leq \frac{-k}{4}$ then there exists no CMC sphere of constant mean curvature H in $E(k, \tau)$. If $k > 0$ then for any $H \neq 0$ there exists a rotationally invariant CMC sphere of constant mean curvature H in $E(k, \tau)$. For every rotationally invariant CMC sphere in $E(k, \tau)$ the first integral (11) vanishes: $J = 0$. The CMC spheres in $E(k, \tau)$ are unique up to isometries.*

Proof. The proof is based on an analysis of a qualitative behavior of the solutions of (10) depending on the values of J and H . Such an analysis is straightforward and it was done for $E(0, \frac{1}{2})$ in [11, 12]; the case of $E(-1, -\frac{1}{2})$ was shown in [4]. The uniqueness of the CMC spheres was proved in [16, 17]. Also, see [18] for the complete proof of Proposition 2. \square

By Proposition 2 we obtain that on a rotationally invariant CMC sphere in $E(k, \tau)$ the following equality holds:

$$\sin \sigma = Hu. \quad (12)$$

REMARK 3. Although there exists no a minimal sphere in $E(k, \tau)$ such spheres exist for $\mathbb{S}^2 \times \mathbb{R}$ and the Berger spheres. We recall that for $k > 0$ the manifolds $E(k, \tau)$ are obtained from the corresponding homogeneous manifolds by removing one fibre.

4 The proof of Theorem 1

For an immersion $f : M \rightarrow E(k, \tau)$ of a closed orientable surface M into $E(k, \tau)$ set:

$$E_{\alpha, \beta}(f) = \int_M (H^2 + \alpha \overline{K} + \beta) d\mu. \quad (13)$$

Let $F : M \times [0, 1] \rightarrow E(k, \tau)$ be a normal variation of the immersion f , i.e., $F(p, 0) = f(p)$ for all $p \in M$ and $\frac{\partial F(p, t)}{\partial t} = \varphi n$, where n is the unit normal vector field to M and the velocity φ is a smooth function on M . We will denote by δ the operator $\frac{\partial}{\partial t}|_{t=0}$. We will need the following proposition.

Proposition 3. *Under a normal variation with the velocity φ the following identities hold:*

$$\delta d\mu = -2H\varphi d\mu, \quad (14)$$

$$\delta n = -\nabla \varphi, \quad (15)$$

$$2\delta H = \Delta \varphi + (4H^2 - 2K_e + \text{Ric}(n, n))\varphi, \quad (16)$$

where ∇ is the gradient and Δ is the Laplace–Beltrami operator on M , K_e is the extrinsic Gauss curvature.

Proof. The proof is standard, one may look it up in [19]. \square

It follows from Proposition 1 that the term $\text{Ric}(n, n)$ equals $k - 2\tau^2 - (k - 4\tau^2)\nu^2$. Therefore we have that:

$$2\delta H = \Delta \varphi + (4H^2 - 2K_e + k - 2\tau^2 - (k - 4\tau^2)\nu^2)\varphi. \quad (17)$$

Let T be the projection of the vertical field ξ on M , i.e., $T = \xi - \nu n$. By (15) we have that $\delta \nu = \delta \langle n, \xi \rangle = -\langle \nabla \varphi, \xi \rangle$. Therefore we obtain:

$$\int_M \nu \delta \nu d\mu = \int_M \text{div}(\nu T) \varphi d\mu. \quad (18)$$

By Proposition 1 we have:

$$E_{\alpha, \beta}(f) = \int_M (H^2 + \alpha(k - 4\tau^2)\nu^2 + \beta + \alpha\tau^2) d\mu. \quad (19)$$

Then by (17),(18) and (14) we obtain:

$$\begin{aligned} \delta E_{\alpha,\beta}(f) = \int_M (\Delta H + H(2H^2 - 2K_e - (1 + 2\alpha)(k - 4\tau^2)\nu^2 + \\ + k - 2\tau^2 - 2\beta - 2\alpha\tau^2) + 2\alpha(k - 4\tau^2)\operatorname{div}(\nu T))\varphi d\mu. \end{aligned} \quad (20)$$

Therefore the Euler–Lagrange equation of the functional (13) is as follows:

$$\begin{aligned} \Delta H + H(2H^2 - 2K_e - (1 + 2\alpha)(k - 4\tau^2)\nu^2 + \\ + k - 2\tau^2 - 2\beta - 2\alpha\tau^2) + 2\alpha(k - 4\tau^2)\operatorname{div}(\nu T) = 0. \end{aligned} \quad (21)$$

By the Gauss theorem we obtain $K_e = K - \overline{K} = K - (k - 4\tau^2)\nu^2 - \tau^2$, where K is the intrinsic Gauss curvature. Then (21) can be rewritten as follows:

$$\begin{aligned} \Delta H + H(2H^2 - 2K + (1 - 2\alpha)(k - 4\tau^2)\nu^2 + \\ + k - 2\beta - \alpha\tau^2) + 2\alpha(k - 4\tau^2)\operatorname{div}(\nu T) = 0. \end{aligned} \quad (22)$$

Consider a CMC sphere in $E(k, \tau)$. For the coordinates on this sphere we choose θ and s ; recall that θ is an angle from the cylindrical coordinate system in $E(k, \tau)$ and s is the natural parameter on the projection $\gamma(s) = (u(s), v(s))$ of this sphere onto $B(k, \tau)$. By (5) for these coordinates the metric on a CMC sphere is as follows:

$$\frac{\rho^2 + \tau^2\rho^4}{(1 + \frac{k}{4}\rho^2)^2}d\theta^2 + ds^2. \quad (23)$$

By (6), (10) and (23) it can be straightforwardly verified that on a CMC sphere in $E(k, \tau)$:

$$\nu = \frac{\cos \sigma}{\sqrt{1 + \tau^2 u^2}}, \quad (24)$$

$$K = -\frac{1 + \frac{k}{4}u^2}{u\sqrt{1 + \tau^2 u^2}} \frac{d}{ds} \frac{u\sqrt{1 + \tau^2 u^2}}{1 + \frac{k}{4}u^2}, \quad (25)$$

$$\operatorname{div}(\nu T) = \frac{1 + \frac{k}{4}u^2}{u\sqrt{1 + \tau^2 u^2}} \frac{d}{ds} \frac{u \cos \sigma \sin \sigma}{(1 + \frac{k}{4}u^2)\sqrt{1 + \tau^2 u^2}}. \quad (26)$$

Put $\alpha = \frac{1}{4}$ and $\beta = \frac{k}{4} - \frac{\tau^2}{4}$. Substituting (24),(25) and (26) into the left hand side of (22) we obtain that it equals:

$$\begin{aligned} 2H^3 + H \left(2 \frac{1 + \frac{k}{4}u^2}{u\sqrt{1 + \tau^2 u^2}} \frac{d}{ds} \frac{u\sqrt{1 + \tau^2 u^2}}{1 + \frac{k}{4}u^2} + \frac{1}{2}(k - 4\tau^2) \frac{\cos^2 \sigma}{1 + \tau^2 u^2} + \frac{k}{2} \right) + \\ + \frac{1}{2}(k - 4\tau^2) \frac{1 + \frac{k}{4}u^2}{u\sqrt{1 + \tau^2 u^2}} \frac{d}{ds} \frac{u \cos \sigma \sin \sigma}{(1 + \frac{k}{4}u^2)\sqrt{1 + \tau^2 u^2}} \end{aligned} \quad (27)$$

Using the equation (12) and the system (10) it can be verified that the expression (27) vanishes on a CMC sphere. Theorem 1 is proved.

5 The proof of Theorem 2

Let us substitute (4) into the formula (2) for the functional $E(f)$. Then we have:

$$E(f) = \int_M \left(H^2 + \left(\frac{k}{4} - \tau^2 \right) \nu^2 + \frac{k}{4} \right) d\mu. \quad (28)$$

Let us consider a rotationally invariant sphere in $E(k, \tau)$ defined by a curve $\gamma(s) = (u(s), v(s)) \subset B(k, \tau)$. By (9) we can represent H^2 as follows:

$$H^2 = \frac{1}{4} \left(\dot{\sigma} - \left(\frac{1}{u} + k \frac{u}{4} \right) \sin \sigma \right)^2 + \left(\frac{\dot{\sigma} \sin \sigma}{u} - k \frac{\sin^2 \sigma}{4} \right). \quad (29)$$

For a rotationally invariant surface we have:

$$\nu^2 = \frac{\cos^2 \sigma}{1 + \tau^2 u^2}, \quad (30)$$

$$d\mu = \frac{u \sqrt{1 + \tau^2 u^2}}{1 + \frac{k}{4} u^2} ds. \quad (31)$$

Substituting (29), (30) and (31) into (28) we obtain:

$$\begin{aligned} E(f) = & 2\pi \int_{\gamma} \frac{1}{4} \left(\dot{\sigma} - \left(\frac{1}{u} + k \frac{u}{4} \right) \sin \sigma \right)^2 \frac{u \sqrt{1 + \tau^2 u^2}}{1 + \frac{k}{4} u^2} ds + \\ & 2\pi \int_{\gamma} \left(\frac{\dot{\sigma} \sin \sigma}{u} - k \frac{\sin^2 \sigma}{4} + \left(\frac{k}{4} - \tau^2 \right) \frac{\cos^2 \sigma}{1 + \tau^2 u^2} + \frac{k}{4} \right) \frac{u \sqrt{1 + \tau^2 u^2}}{1 + \frac{k}{4} u^2} ds. \end{aligned} \quad (32)$$

By (10) we have: $\cos \sigma = \frac{\dot{u}}{1 + \frac{k}{4} u^2}$. Substituting this into the integrand of the second summand in (32) we obtain:

$$- \frac{\ddot{u} \sqrt{1 + \tau^2 u^2}}{(1 + \frac{k}{4} u^2)^2} + \frac{u \dot{u}^2}{(1 + \frac{k}{4} u^2)^3} \left(\frac{3}{4} k \sqrt{1 + \tau^2 u^2} + \left(\frac{k}{4} - \tau^2 \right) \frac{1}{\sqrt{1 + \tau^2 u^2}} \right). \quad (33)$$

It can be verified that the expression (33) is equal to $\frac{d}{ds} \left[-\frac{\dot{u} \sqrt{1 + \tau^2 u^2}}{(1 + \frac{k}{4} u^2)^2} \right]$. Therefore, for a rotationally invariant sphere the second summand in (32) is equal to 4π .

The first summand in (32) is nonnegative. It vanishes iff the following holds:

$$\dot{\sigma} - \left(\frac{1}{u} + k \frac{u}{4} \right) \sin \sigma = 0. \quad (34)$$

It follows from (10) that (34) holds iff a rotationally invariant sphere is CMC. Theorem 2 is proved.

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